

# Vertex operator superalgebra structure for degenerate minimal models: Neveu-Schwarz algebra

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## Abstract

The  $\mathbb{Z}/2\mathbb{Z}$ -graded intertwining operators are introduced. We study these operators in the case of “degenerate”  $N = 1$  minimal models, with the central charge  $c = \frac{3}{2}$ . The corresponding fusion ring is isomorphic to the Grothendieck ring for the Lie superalgebra  $\mathfrak{osp}(1|2)$ . We also discuss multiplicity-2 fusion rules and logarithmic intertwiners.

## 1 Introduction

This paper is a continuation of [M1]. It is also closely related to [HM]. For a more detailed introduction see [M1].

We introduce the notion of a  $\mathbb{Z}/2\mathbb{Z}$ -graded (*even* and *odd*) intertwining operator and use it to study the *fusion ring* for the “degenerate” ( $p = q$ ) minimal models, for the vertex operator superalgebra  $L(\frac{3}{2}, 0)$ . As in [M1] there are two approaches: one which uses the lattice construction (extended with a suitable fermionic Fock space) and the other which uses the singular vectors and projection formulas. For the future purposes we use the latter approach.

The degenerate minimal models are irreducible modules for the  $N = 1$  superconformal vertex operator algebra  $L(\frac{3}{2}, 0)$  (cf. [KW], [A]) that are isomorphic to  $L(\frac{3}{2}, \frac{q^2}{2})$ ,  $q \in \mathbb{N}$ . We prove (see Theorem 7.1 and Corollary 7.1) that the corresponding fusion ring is isomorphic to the Grothendieck ring for  $\mathfrak{osp}(1|2)$ , i.e., we formally have:

$$L\left(\frac{3}{2}, \frac{r^2}{2}\right) \times L\left(\frac{3}{2}, \frac{q^2}{2}\right) = \\ L\left(\frac{3}{2}, \frac{(r+q)^2}{2}\right) + L\left(\frac{3}{2}, \frac{(r+q-1)^2}{2}\right) \dots + L\left(\frac{3}{2}, \frac{(r-q)^2}{2}\right),$$

for every  $r, q \in \mathbb{N}$ ,  $r \geq q$ .

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As in the Virasoro algebra case, these fusion coefficients are 0 or 1. However in [HM] we showed that for some vertex operator algebras  $L(c, 0)$ , fusion coefficients might be 2. In Proposition 8.1 we construct a non-trivial example when  $c = \frac{15}{2} - 3\sqrt{5}$ .

At the very end, we construct an example of a *logarithmic intertwining operator* (for the definition see [M2]) for the  $N = 1$  vertex operator superalgebra  $L(\frac{27}{2}, 0)$  (cf. Proposition 8.2).

## 2 Lie superalgebra $\mathfrak{osp}(1|2)$ and $\mathcal{Rep}(\mathfrak{osp}(2|1))$

The Lie superalgebra  $\mathfrak{osp}(1|2)$  is a graded extension of the finite-dimensional Lie algebra  $\mathfrak{sl}(2, \mathbb{C})$ . It has three even generators  $x, y$  and  $h$ , and two odd generators  $\varphi$  and  $\chi$ , that satisfy

$$\begin{aligned} [h, x] &= 2x, \quad [h, y] = -2y, \quad [x, y] = h, \\ [x, \chi] &= \chi, \quad [x, \varphi] = -\varphi, \quad [y, \chi] = -\chi, \quad [y, \varphi] = \varphi, \\ [h, \varphi] &= -\varphi, \quad [h, \chi] = \chi, \\ \{\chi, \varphi\} &= 2h, \quad \{\chi, \chi\} = 2x, \quad \{\varphi, \varphi\} = 2y. \end{aligned}$$

Generators  $\{x, y, h\}$  span a Lie algebra isomorphic to  $\mathfrak{sl}(2, \mathbb{C})$ , and this fact makes the representation theory of  $\mathfrak{osp}(1|2)$  quite simple. All irreducible  $\mathfrak{osp}(1|2)$ -modules can be constructed in the following way. Fix a positive half integer  $j$  ( $2j \in \mathbb{N}$ ) and a  $4j + 1$ -dimensional vector space  $V(j)$  spanned by the vectors  $\{v_j, v_{j-1/2}, \dots, v_{-j}\}$ , with the following actions:

$$\begin{aligned} x.v_i &= \sqrt{[j-i][j+i+1]}v_{i+1}, \\ y.v_i &= \sqrt{[j+i][j-i+1]}v_{i-1}, \\ h.v_i &= 2iv_i. \end{aligned} \tag{1}$$

If  $2(i-j) \in \mathbb{Z}$  then we define

$$\begin{aligned} \varphi.v_i &= -\sqrt{j+i}v_{i-1/2}, \\ \chi.v_i &= -\sqrt{j-i}v_{i+1/2}, \end{aligned} \tag{2}$$

otherwise

$$\begin{aligned} \varphi.v_i &= \sqrt{j-i+1/2}v_{i-1/2}, \\ \chi.v_i &= -\sqrt{j+i+1/2}v_{i+1/2}. \end{aligned} \tag{3}$$

In all these formulas  $v_j = 0$  if  $j \notin \{j, j - \frac{1}{2}, \dots, -j\}$ . It is easy to see that each  $V(j)$  is an irreducible  $\mathfrak{osp}(1|2)$ -module and that every finite dimensional irreducible representation of  $\mathfrak{osp}(1|2)$  is isomorphic to some  $V(j)$  for  $j \in \mathbb{N}/2$ .

It is a pleasant exercise to decompose tensor product  $V(i) \otimes V(j)$ . The following result is well-known (see [FM] for instance):

$$V(i) \otimes V(j) \cong \bigoplus_{k=|i-j|, k \in \mathbb{N}/2}^{i+j} V(k). \tag{4}$$

### 3 $N = 1$ Neveu-Schwarz superalgebra and its minimal models

The  $N = 1$  Neveu-Schwarz superalgebra is given by

$$\mathfrak{ns} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} L_n \bigoplus \bigoplus_{n \in \mathbb{Z}} \mathbb{C} G_{n+1/2} \bigoplus \mathbb{C} C,$$

together with the following  $N = 1$  Neveu-Schwarz relations:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n,0}, \\ [L_m, G_{n+1/2}] &= \left( \frac{m}{2} - \left( n + \frac{1}{2} \right) \right) G_{m+n+1/2}, \\ [G_{m+1/2}, G_{n-1/2}] &= 2L_{m+n} + \frac{C}{3}(m^2 + m)\delta_{m+n,0}, \\ [C, L_m] &= 0, \\ [C, G_{m+1/2}] &= 0 \end{aligned}$$

for  $m, n \in \mathbb{Z}$ . We have the standard triangular decomposition  $\mathfrak{ns} = \mathfrak{ns}_+ \oplus \mathfrak{ns}_0 \oplus \mathfrak{ns}_-$  (cf. [HM]). For every  $(h, c) \in \mathbf{C}^2$ , we denote by  $M(c, h)$  Verma module for  $\mathfrak{ns}$  algebra. For each  $(p, q) \in \mathbf{N}^2$ ,  $p = q \bmod 2$ , let us introduce a family of complex 'curves'  $(h_{p,q}(t), c(t))$ ;

$$\begin{aligned} h_{p,q}(t) &= \frac{1-p^2}{8}t^{-1} + \frac{1-pq}{4} + \frac{1-q^2}{8}t, \\ c(t) &= \frac{15}{2} + 3t^{-1} + 3t. \end{aligned}$$

Then from the determinant formula (see [KWa]) it follows that  $M(c, h)$  is reducible if and only if there is a  $t \in \mathbf{C}$  and  $p, q \in \mathbf{N}$ ,  $p = q \bmod 2$  such that  $c = c(t)$  and  $h = h_{p,q}(t)$ . In this case  $M(c, h)$  has a *singular vector* (i.e., a vector annihilated by  $\mathfrak{ns}_+$ ) of the weight  $h + \frac{pq}{2}$ . Any such vector we denote by  $v_{\frac{pq}{2}}$ .

In this paper we are interested in the  $t = -1$ . Then  $c(-1) = \frac{3}{2}$  and  $h_{p,q}(-1) = \frac{(p-q)^2}{8}$ .  $h_{p,q}(-1) = h_{1,p-q+1}(-1)$ , so we consider only the case  $h_{1,q} := h_{1,q}(-1)$ , (here  $q$  is odd and positive). Hence, each Verma module  $M(\frac{3}{2}, h_{1,q})$  is reducible.

The following result easily follows from [D] (or [AA]) and [KWa]:

**Proposition 3.1** *For every odd  $q$ ,  $M(\frac{3}{2}, h_{1,q})$  has the following embedding structure*

$$\dots \rightarrow M\left(\frac{3}{2}, h_{1,q+4}\right) \rightarrow M\left(\frac{3}{2}, h_{1,q+2}\right) \rightarrow M\left(\frac{3}{2}, h_{1,q}\right) \rightarrow 0. \quad (5)$$

Moreover, we have the following exact sequence:

$$0 \rightarrow M\left(\frac{3}{2}, h_{1,q+2}\right) \rightarrow M\left(\frac{3}{2}, h_{1,q}\right) \rightarrow L\left(\frac{3}{2}, h_{1,q}\right) \rightarrow 0, \quad (6)$$

where  $L(\frac{3}{2}, h_{1,q})$  is the corresponding irreducible quotient. ■

Benoit and Saint-Aubin (cf. [BSA]) found an explicit expression for the singular vector  $v_{1,q} \in M(\frac{3}{2}, h_{1,q})$  that generates the maximal submodule:

$$v_{1,q} = \sum_{N; k_1, \dots, k_N} \sum_{\sigma \in S_N} (-1)^{\frac{q-N}{2}} c(k_{\sigma(1)}, \dots, k_{\sigma(k)}) G_{-k_1/2} \dots G_{-k_N/2} v, \quad (7)$$

where  $S_N$  is a symmetric group on  $N$  letters and the first summation is over all the partitions of  $q$  into the odd integers  $k_1, \dots, k_N$  and

$$c(k_{\sigma(1)}, \dots, k_{\sigma(k)}) = \prod_{i=1}^N \binom{k_i - 1}{(k_i - 1)/2} \prod_{j=1}^{(N-1)/2} \frac{4}{\sigma_{2j} \rho_{2j}},$$

where  $\sigma_j = \sum_{l=1}^j k_l$  and  $\rho_j = \sum_{l=j}^N k_l$ .

In the special case:  $q = 1$ ,  $h_{1,1} = 0$ ,  $M(\frac{3}{2}, 0)$  has a singular vector  $G(-1/2)v$  which generate the maximal submodule. By quotienting we obtain a *vacuum* module  $L(\frac{3}{2}, 0) = M(\frac{3}{2}, 0) / \langle G_{-1/2} v_{3/2,0} \rangle$ .

## 4 $N = 1$ superconformal vertex operator superalgebra and intertwining operators

We use the definition of  $N = 1$  superconformal vertex operator superalgebra (with and without odd variables) as in [B] and [HM] (see also [KW] and [KV]).

Let  $\varphi$  be a Grassman (odd) variable such that  $\varphi^2 = 0$ . Every  $N = 1$  superconformal vertex operator superalgebra  $(V, Y, \mathbf{1}, \tau)$  can be equipped with a structure of  $N = 1$  superconformal vertex operator algebra with odd variables via

$$\begin{aligned} Y(\cdot, (x, \varphi)) : V \otimes V &\rightarrow V((x))[\varphi], \\ u \otimes v &\mapsto Y(u, (x, \varphi))v, \end{aligned}$$

where

$$Y(u, (x, \varphi))v = Y(u, x)v + \varphi Y(G(-1/2)u, x)v$$

for  $u, v \in V$ .

The same formula can be used in the case of modules for the superconformal vertex operator superalgebra  $(V, Y, \mathbf{1}, \tau)$  (see [HM]).

It is known ([KW]) that  $V(c, 0) := M(c, 0) / \langle G_{-1/2} v_{c,0} \rangle$ <sup>1</sup> is a  $N = 1$  superconformal vertex operator superalgebra. Also, every lowest weight  $\mathfrak{ns}$ -module with the central charge  $c$ , is a  $V(c, 0)$ -module. If  $c = \frac{3}{2}$  then  $V(\frac{3}{2}, 0) = L(\frac{3}{2}, 0)$ . Hence

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<sup>1</sup>We write  $L(c, 0)$  if  $V(c, 0)$  is irreducible.

**Proposition 4.1** *Every irreducible  $L(\frac{3}{2}, 0)$ -module is isomorphic to  $L(\frac{3}{2}, h)$ , for some  $h \in \mathbb{C}$ .*

*Proof:* It is known (cf. [KW]) that there is one-to-one correspondence between equivalence classes of irreducible  $A(L(\frac{3}{2}, 0))$ -modules (here  $A(L(\frac{3}{2}, 0)) \cong \mathbb{C}[y]$  is Zhu's associative algebra) and irreducible  $L(\frac{3}{2}, 0)$ -modules. If  $W$  is an irreducible  $A(L(\frac{3}{2}, 0))$ -module, then there is an irreducible  $L(\frac{3}{2}, 0)$ -module  $\Omega(W)$  (that is  $\mathbb{N}$ -gradable) such that  $\Omega(W)(0) \cong W$ . Because  $\Omega(W)$  is a  $\mathfrak{ns}$ -irreducible module it is also  $\mathbb{N}$ -gradable. Therefore  $\Omega(W) \cong L(c, h)$  for some  $h \in \mathbb{C}$ . ■

Among all irreducible  $L(\frac{3}{2}, 0)$ -modules we distinguish modules isomorphic to  $L(\frac{3}{2}, h_{1,q})$ ,  $q \in 2\mathbb{N} - 1$ . These are so called degenerate minimal models.

#### 4.1 Intertwining operators and its matrix coefficients

The notation of an intertwining operators for  $N = 1$  superconformal vertex operator algebras is introduced in [KW] and [HM].

Let  $W_1, W_2$  and  $W_3$  be a triple of  $V$ -modules and  $\mathcal{Y}$  an intertwining operator of type  $(\begin{smallmatrix} W_3 \\ W_1 W_2 \end{smallmatrix})$ . Then we consider the corresponding intertwining operator with odd variable (cf. [HM]):

$$\begin{aligned} \mathcal{Y}(\cdot, (x, \varphi)) : W_1 \otimes W_2 &\rightarrow W_3\{x\}[\varphi] \\ w_{(1)} \otimes w_{(2)} &\mapsto \mathcal{Y}(w_{(1)}, (x, \varphi))w_{(2)}, \end{aligned}$$

such that

$$\mathcal{Y}(w_{(1)}, (x, \varphi))w_{(2)} = \mathcal{Y}(w_{(1)}, x)w_{(2)} + \varphi \mathcal{Y}(G(-1/2)w_{(1)}, x)w_{(2)}.$$

Let  $w_1$  be a lowest weight vector for the Neveu-Schwarz algebra of the weight  $h$ . From the Jacobi identity we derive the following formulas:

$$\begin{aligned} [L(-n), \mathcal{Y}(w_1, x_2)] &= (x_2^{-n+1} \frac{\partial}{\partial x_2} + (1-n)h) \mathcal{Y}(w_1, x_2), \\ [G(-n-1/2), \mathcal{Y}(w_1, x_2)] &= x_2^{-n} \mathcal{Y}(G(-1/2)w_1, x_2), \\ [L(-n), \mathcal{Y}(G(-1/2)w_1, x_2)] &= (x_2^{-n+1} \frac{\partial}{\partial x_2} + (1-n)(h + \frac{1}{2})) \mathcal{Y}(G(-1/2)w_1, x_2), \\ [G(-n-1/2), \mathcal{Y}(G(-1/2)w_1, x_2)] &= (x_2^{-n} \frac{\partial}{\partial x_2} - 2nhx_2^{-n-1}) \mathcal{Y}(w_1, x_2). \end{aligned} \quad (8)$$

In the odd formulation we obtain

$$\begin{aligned} &[L(-n), \mathcal{Y}(w_1, (x_2, \varphi))] \\ &= (x_2^{-n+1} \partial_{x_2} + (1-n)x_2^{-n}(h + 1/2\varphi\partial_\varphi)) \mathcal{Y}(w_1, (x_2, \varphi)) \\ &[G(-n-1/2), \mathcal{Y}(w_1, (x_2, \varphi))] \\ &= (x_2^{-n}(\partial_\varphi - \varphi\partial_{x_2}) - 2nx_2^{-n-1}(h\varphi)) \mathcal{Y}(w_1, (x_2, \varphi)), \end{aligned} \quad (9)$$

where  $\partial_\varphi$  is the odd (Grassmann) derivative.

## 4.2 Even and odd intertwining operators

In [HM] we proved that every intertwining operator

$$\mathcal{Y} \in I \left( \begin{matrix} L(c, h_3) \\ L(c, h_1) \ L(c, h_2) \end{matrix} \right)$$

is uniquely determined by the operators  $\mathcal{Y}(w_1, x)$  and  $\mathcal{Y}(G(-1/2)w_1, x)$ , where  $w_1$  is the highest weight vector of  $L(c, h_1)$ . This fact will be used later in connection with the following definition.

**Definition 4.1** Let  $| \cdot |$  denote the parity (0 or 1). An intertwining operator  $\mathcal{Y} \in I \left( \begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix} \right)$  is:

- *even*, if

$$\text{Coeff}_{x^s} |\mathcal{Y}(w_1, x)w_2| = |w_1| + |w_2|,$$

- *odd*, if

$$\text{Coeff}_{x^s} |\mathcal{Y}(w_1, x)w_2| = |w_1| + |w_2| + 1,$$

for every  $s \in \mathbb{C}$  and every  $\mathbb{Z}/2\mathbb{Z}$ -homogeneous vectors  $w_1$  and  $w_2$ .

The space of even (odd) intertwining operators of the type  $\mathcal{Y} \in I \left( \begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix} \right)$  we denote by  $I \left( \begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix} \right)_{\text{even}}$  ( $I \left( \begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix} \right)_{\text{odd}}$ ). In general one does not expect a decomposition of  $I \left( \begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix} \right)$  into the even and the odd subspace.

## 4.3 Frenkel-Zhu's theorem for vertex operator superalgebras

According to [KW] (after [Z]), to every vertex operator superalgebra we can associate Zhu's associative algebra  $A(V)$ . If  $V = L(c, 0)$ ,  $A(L(c, 0)) \cong \mathbb{C}[y]$ , where  $y = [(L(-2) - L(-1))\mathbf{1}] = [L(-2)\mathbf{1}]$  (because of the calculations that follow it is convenient to use  $y = [(L(-2) - L(-1))\mathbf{1}]$ ). Also to every  $V$ -module  $W$  we associate a  $A(V)$ -bimodule  $A(W)$  (cf. [KW]). In a special case  $W = M(c, h)$ , we have

$$A(M_{\text{ns}}(c, h)) = M_{\text{ns}}(c, h)/O(M_{\text{ns}}(c, h)),$$

where

$$\begin{aligned} O(M_{\text{ns}}(c, h)) &= \{L(-n-3) - 2L(-n-2) + L(-1)v, \\ G(-n-1/2) - G(-n-3/2)v, n \geq 0, v \in M(c, h)\}. \end{aligned} \quad (10)$$

It is not hard to see that, as  $\mathbb{C}[y]$ -bimodule,

$$A(M(c, h)) \cong \mathbb{C}[x, y] \oplus \mathbb{C}[x, y]v,$$

where  $v = [G(-1/2)v_h]$  and

$$y = [L(-2) - L(-1)], \quad x = [L(-2) - 2L(-1) + L(0)].$$

Let  $W_1$ ,  $W_2$  and  $W_3$  be three  $\mathbb{N}/2$ -gradable irreducible  $V$ -modules such that  $\text{Spec} L(0)|_{W_i} \in h_i + \mathbb{N}$ ,  $i = 1, 2, 3$  and  $\mathcal{Y} \in I \left( \begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix} \right)$ . We define  $o(w_1) := \text{Coeff}_{x^{h_3-h_1-h_2}} \mathcal{Y}(w_1, x)$ . Because the fusion rules formula in [FZ] needs some modifications (cf. [L1]) the same modification is necessary for the main Theorem in [KW] (this can be done with a minor super-modifications along the lines of [L1]). Nevertheless (cf. [KW]):

**Theorem 4.2** *The mapping*

$$\pi : I \left( \begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix} \right) \rightarrow \text{Hom}_{A(V)}(A(W_1) \otimes_{A(V)} W_2(0), W_3(0)),$$

*such that*

$$\pi(\mathcal{Y})(w_1 \otimes w_2) = o(w_1)w_2, \quad (11)$$

*is injective.*

## 5 Some Lie superalgebra homology

In this section we recall some basic definition from the homology theory of infinite dimensional Lie superalgebras which is in the scope of the monograph [F] (in the cohomology setting though).

Let  $\mathcal{L}$  be an any (possibly infinite dimensional)  $\mathbb{Z}/2\mathbb{Z}$ -graded Lie superalgebra with the  $\mathbb{Z}/2\mathbb{Z}$ -decomposition:  $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$ . and let  $M = M_0 \oplus M_1$  be any  $\mathbb{Z}_2$ -graded  $\mathcal{L}$ -module, such that the gradings are compatible. Then, we form a chain complex  $(C, d, M)$  (for details see [F]),

$$0 \xleftarrow{d_0} C_0(\mathcal{L}, M) \xleftarrow{d_1} C_1(\mathcal{L}, M) \xleftarrow{d} \dots,$$

where

$$C_q(\mathcal{L}, M) = \bigoplus_{q_0+q_1=q} M \otimes \Lambda^{q_0} \mathcal{L}_0 \otimes S^{q_1} \mathcal{L}_1,$$

$$C_q^p(\mathcal{L}, M) = \bigoplus_{\substack{q_0+q_1=q \\ q_1+r=p \bmod 2}} M_r \otimes \Lambda^{q_0} \mathcal{L}_0 \otimes S^{q_1} \mathcal{L}_1,$$

for  $p = 0, 1$ . The mappings  $d$  are super-differentials. For  $q \in \mathbb{N}$  and  $p = 0, 1$ , we define  $q$ -th homology with coefficients in  $M$  as:

$$H_q^p(\mathcal{L}, M) = \text{Ker}(d_q(C_q^p(\mathcal{L}, M)))_p / (d_{q+1}(C_{q+1}^p(\mathcal{L}, M)))_p. \quad (12)$$

In a special case  $q = 0$ , we have

$$H_0^0(\mathcal{L}, M) = M_0 / (\mathcal{L}_0 M_0 + \mathcal{L}_1 M_1),$$

and

$$H_0^1(\mathcal{L}, M) = M_1 / (\mathcal{L}_1 M_0 + \mathcal{L}_0 M_1).$$

We want to calculate  $H_q(\mathcal{L}_s, L(\frac{3}{2}, h_{1,q}))$ . for the Lie superalgebra

$$\mathcal{L}_s = \bigoplus_{n \geq 0} \mathcal{L}_s(n),$$

where  $\mathcal{L}_s(n)$  is spanned by the vectors  $L(-n-3) - 2L(-n-2) + L(-n-1)$  and  $G(-n-1/2) - G(-n-3/2)$ ,  $n \in \mathbf{N}$ . From (10) we see (cf. [HM]) that  $H_0(\mathcal{L}_s, M(c, h))$  is a  $\mathbb{C}[y]$ -bimodule such that:

$$H_0(\mathcal{L}_s, M(c, h)) \cong A(M(c, h)) \cong \mathbb{C}[x, y] \oplus \mathbb{C}[x, y]v. \quad (13)$$

**Remark 5.1** It is more involved to calculate  $H_0((\mathcal{L}_s, L(c, h)))$ , so we consider only the special case  $c = \frac{3}{2}$ ,  $h = h_{1,q}$ ,  $q$  odd. As in the Virasoro case (see [M1]) it is easy to show that the space  $H_p(\mathcal{L}_s, L(\frac{3}{2}, h_{1,q}))$  is infinite dimensional for very  $p, q, s \in \mathbf{N}$ , and finitely generated as a  $A(L(3/2, 0))$ -module. Moreover,

$$\text{Ext}^1(L(\frac{3}{2}, h_{1,q}), L(\frac{3}{2}, h_{1,r}))$$

is one-dimensional if  $r = q + 2$  and 0 otherwise. We will not need these results.

In the minimal models case we expect a substantially different result (cf [FF1]).

**Conjecture 5.1** Let  $c_{p,q} = \frac{3}{2} \left(1 - 2 \frac{(p-q)^2}{pq}\right)$  and  $h_{p,q}^{m,n} = \frac{(np-mq)^2 - (p-q)^2}{8pq}$ . Then

$$\dim H_q(\mathcal{L}_s, L(c_{p,q}, h_{p,q}^{m,n})) < \infty,$$

for every  $q \in \mathbf{N}$ .

There is strong evidence that Conjecture (5.1) holds based on [A] and an example  $c = -\frac{11}{14}$  treated in Appendix of [HM].

The main difference between the minimal models and the degenerate models is the fact that the maximal submodule for a minimal model is generated by two singular vectors, compared to  $M(\frac{3}{2}, h_{1,q})$  where the maximal submodule is generated by a single singular vector.

## 6 Benoit-Saint-Aubin's formula projection formulas

### 6.1 Odd variable formulation

We have seen before how to derive the commutation relation between generators of  $\mathfrak{ns}$  superalgebra and  $\mathcal{Y}(w_1, x)$  where  $w_1$  is a lowest weight vector for  $\mathfrak{ns}$ . We fix  $\mathcal{Y} \in I_{L(\frac{3}{2}, h_{1,r})}^{L(\frac{3}{2}, h)} L(\frac{3}{2}, h_{1,q})$  and consider the following matrix coefficient,

$$\langle w'_3, \mathcal{Y}(w_1, x, \varphi) P_{\text{sing}} w_2 \rangle, \quad (14)$$



where  $P_{\text{sing}}w_2 = v_{1,q}$  ( $\deg(P_{\text{sing}}) = q/2$ ) and  $w_i$ ,  $i = 1, 2, 3$  are the lowest weight vectors.

Since all modules are irreducible, by using a result from [HM] (Proposition 2.2), we get

$$\langle w'_3, \mathcal{Y}(w_1, x, \varphi)w_2 \rangle = c_1 x^{h-h_{1,q}-h_{1,r}} + c_2 \varphi x^{h-h_{1,q}-h_{1,r}-1/2},$$

where  $c_1$  and  $c_2$  are constants with the property

$$c_1 = c_2 = 0 \text{ implies } \mathcal{Y} = 0. \quad (15)$$

From the formulas (9)

$$\langle w'_3, \mathcal{Y}(w_1, x, \varphi)P_{\text{sing}}w_2 \rangle = P(\partial_{x_2}, \varphi) \langle w'_3, \mathcal{Y}(w_1, x, \varphi)w_2 \rangle,$$

where  $P(\partial_{x_2}, \varphi)$  is a certain super-differential operator such that

$$\deg(P_{\text{sing}}) = \deg P(\partial_{x_2}, \varphi) = q/2.$$

Therefore

$$P(\partial_{x_2}, \varphi)c_1 x^{h-h_{1,q}-h_{1,r}} = \varphi C_1(h_{1,q}, h_{1,r}, h) x^{h-h_{1,q}-h_{1,r}-q/2},$$

and

$$P(\partial_{x_2}, \varphi)c_2 x^{h-h_{1,q}-h_{1,r}-q/2} = C_2(h_{1,q}, h_{1,r}, h) x^{h-h_{1,q}-h_{1,r}-q/2}.$$

Constants  $C_1(h_{1,q}, h_{1,r}, h)$  and  $C_2(h_{1,q}, h_{1,r}, h)$  (in slightly different form, but in more general setting) were derived in [BSA]. Considering these coefficients was motivated by deriving formulas for singular vectors from already known singular vectors. By slightly modifying result from [BSA] we obtain

**Proposition 6.1** *Suppose that  $\mathcal{Y} \in I_{(L(\frac{3}{2}, h_{1,r}), L(\frac{3}{2}, h_{1,q}))}^{L(\frac{3}{2}, h)}$  and  $P(\partial_x, \varphi)$  are as the above. Then, up to a multiplicative constant,*

$$C_1(h_{1,q}, h_{1,r}, h) = \prod_{-j \leq k \leq j} (h - h_{1,q+4k})$$

and

$$C_2(h_{1,q}, h_{1,r}, h) = \prod_{-j+1/2 \leq k \leq j-1/2} (h + \frac{1}{2} - h_{1,q+4k}),$$

for  $j = (r-1)/4$ ,  $j > 0$  (when  $j = 0$ ,  $C_2(h_{1,1}, h_{1,r}, h) = 1$ ).

*Proof:* The superdifferential operator  $P(\partial_x, \varphi)$  is obtained by replacing generators  $L(-m)$  and  $G(-n-1/2)$  by the superdifferential operators

$$L(-m) \mapsto -(x_2^{-m+1} \partial_{x_2} + (1-m)x_2^{-m}(h_1 + 1/2 \varphi \partial_\varphi)) \quad (16)$$

and

$$G(-n-1/2) \mapsto (x_2^{-n}(\partial_\varphi - \varphi \partial_{x_2}) - 2nx_2^{-n-1}(h_1 \varphi)), \quad (17)$$

acting on  $\langle w'_3, \mathcal{Y}(w_1, x, \varphi)w_2 \rangle$ . This action was calculated in [BSA]. Their results (Formula 3.10 in [BSA]) implies the statement <sup>2</sup>.  $\blacksquare$

<sup>2</sup>In [BSA] a different sign was used in the equation (16). Still, we obtain the same result if we consider an isomorphic algebra with the generators  $\tilde{L}(n) := -L(n)$ . The same generators were used in [FF2].

## 6.2 BSA formula without odd variables

Since Frenkel-Zhu's formula does not involve odd variables we need a version of Proposition 6.1 without odd variables (which is of course equivalent). Again  $\mathcal{Y} \in I \left( \begin{smallmatrix} L(3/2, h) \\ L(3/2, h_{1,r}) \quad L(3/2, h_{1,q}) \end{smallmatrix} \right)$  is the same as the above. Then

$$\langle w'_3, \mathcal{Y}(w_1, x) P_{sing} w_2 \rangle = P_2(\partial_x) \langle w'_3, \mathcal{Y}(G(-1/2)w_1, x) w_2 \rangle,$$

and

$$\langle w'_3, \mathcal{Y}(G(-1/2)w_1, x) P_{sing} w_2 \rangle = P_1(\partial_x) \langle w'_3, \mathcal{Y}(w_1, x) w_2 \rangle,$$

where  $P_1$  and  $P_2$  are certain differential operators. If

$$P_2(\partial_x) c_2 x^{h-h_{1,q}-h_{1,r}-1/2} = c_2 K_2(h_{1,q}, h_{1,r}, h) x^{h-h_{1,q}-h_{1,r}-q/2},$$

and

$$P_1(\partial_x) c_1 x^{h-h_{1,q}-h_{1,r}} = c_1 K_1(h_{1,q}, h_{1,r}, h) x^{h-h_{1,q}-h_{1,r}-q/2},$$

then, by comparing corresponding coefficients, we obtain

$$\begin{aligned} K_1(h_{1,q}, h_{1,r}, h) &= C_1(h_{1,q}, h_{1,r}, h), \\ K_2(h_{1,q}, h_{1,r}, h) &= C_2(h_{1,q}, h_{1,r}, h). \end{aligned} \tag{18}$$

Let us mention that the projection formulas from Proposition 6.1 have a simple explanation in the term of *super density modules* for the Neveu-Schwarz superalgebra.

## 7 Fusion ring for the degenerate minimal models

In order to obtain an upper bound for the fusion coefficients (cf. Theorem 4.2) we first compute

$$A(L(\frac{3}{2}, h_{1,q})) \otimes_{A(L(3/2,0))} L(\frac{3}{2}, h_{1,r})(0).$$

$\mathbb{Z}_2$ -grading of the 0-th homology group (12) will enable us (see Theorem (7.1) to study odd and even intertwining operators (see Definition 4.1). For that purpose we introduce the following splitting:

$$\begin{aligned} A^0(L(\frac{3}{2}, h_{1,q})) &:= H_0^0(\mathcal{L}_s, L(\frac{3}{2}, h_{1,q})) \cong \frac{\mathbf{C}[x, y]}{I_1} \\ A^1(L(\frac{3}{2}, h_{1,q})) &:= H_0^1(\mathcal{L}_s, L(\frac{3}{2}, h_{1,q})) \cong \frac{\mathbf{C}[x, y]v}{I_2}, \end{aligned} \tag{19}$$

where  $I_1$  and  $I_2$  are cyclic submodules (the maximal submodule for  $M(3/2, h_{1,q})$  is cyclic !). It seems hard to obtain explicitly these polynomials. First we obtain

some useful formulas Inside  $A(M(c, h))$  (cf. [W]):

$$\begin{aligned} [L(-n)v] &= [(n-1)(L(-2) - L(-1)) + L(-1))v] = \\ &= [(n(L(-2) - L(-1)) - (L(-2) - 2L(-1) + L(0)) + L(0))v] = \\ &= (ny - x + \text{wt}(v))[v]. \end{aligned} \quad (20)$$

for every  $n \in \mathbf{N}$  and every homogeneous  $v \in M(c, h)$ . Therefore in

$$A(M(\frac{3}{2}, h_{1,q})) \otimes_{A(L(\frac{3}{2}, 0))} L(\frac{3}{2}, h_{1,r})(0)$$

we have

$$\begin{aligned} [L(-n)v] &= (nh_{1,q} - x + L(0))[v]. \\ [G(-n - 1/2)v] &= [G(-1/2)v]. \end{aligned} \quad (21)$$

Also, we have:

$$\begin{aligned} [G(-n - \frac{1}{2})G(-m - \frac{1}{2})v] &= [G(-1/2)G(-m - 1/2)v] = \\ &= [(2L(-m - 1) - G(-m - 1/2)G(-1/2))v] = [(2L(-m - 1) - L(-1))v] = \\ &= ((2m + 1)y - x + \text{wt}(v))[v]. \end{aligned} \quad (22)$$

By using (20) and (22) we obtain

$$\begin{aligned} &[G(-m_1 - 1/2) \dots G(-m_{2r} - 1/2)L(-n_1) \dots L(-n_s)v_{1,q}] = \\ &\prod_{i=1}^r ((2m_{2i} + 1)h_{1,r} - x + \sum_{p=2i+1}^{2r} (m_p + 1/2) + h_{1,q}) \cdot \\ &\prod_{j=1}^s (n_j h_{1,r} - x + \sum_{p=j+1}^s n_p + h_{1,q})[v]. \end{aligned} \quad (23)$$

inside

$$A(M(\frac{3}{2}, h_{1,q})) \otimes_{A(L(\frac{3}{2}, 0))} L(\frac{3}{2}, h_{1,r})(0).$$

It is easy to obtain a similar formula for the vector

$$[G(-m_1 - 1/2) \dots G(-m_{2r+1} - 1/2)L(-n_1) \dots L(-n_s)v_{1,q}].$$

**Lemma 7.1** *Let  $[P_{\text{sing}}v_{1,q}] = Q_1(x)[G(-1/2)v_{1,q}]$  and  $[G(-1/2)P_{\text{sing}}v_{1,q}] = Q_2(x)[v_{1,q}]$  be projections inside*

$$A(M(\frac{3}{2}, h_{1,q})) \otimes_{A(L(\frac{3}{2}, 0))} L(\frac{3}{2}, h_{1,r})(0).$$

*Then*

$$\begin{aligned} Q_1(h) &= K_2(h_{1,q}, h_{1,r}, h), \\ Q_2(h) &= K_1(h_{1,q}, h_{1,r}, h), \end{aligned} \quad (24)$$

*for every  $h \in \mathbb{C}$ .*

*Proof:* We use the notation from the section 6.2, where

$$\mathcal{Y} \in I \left( \begin{array}{c} L(3/2, h) \\ L(3/2, h_{1,r}) \ L(3/2, h_{1,q}) \end{array} \right).$$

By using (8), we obtain

$$\begin{aligned} & \langle w'_3, \mathcal{Y}(w_1, x) G(-m_1 - 1/2) \dots G(-m_{2r} - 1/2) L(-n_1) \dots L(-n_s) w_2 \rangle = \\ & \prod_{i=1}^r - (x^{-m_{2i-1}-m_{2i}} \frac{\partial}{\partial x} - 2m_{2i} h_{1,r} x^{-m_{2i-1}-m_{2i}-1}) \cdot \\ & \prod_{j=1}^s - (x^{-n_j+1} \frac{\partial}{\partial x} + (1-n_j) h_{1,r} x^{-n_j}) \langle w'_3, \mathcal{Y}(w_1, x) w_2 \rangle = \\ & c_1 \prod_{i=1}^r ((2m_{2i} + 1) h_{1,r} - h + h_{1,q} + \sum_{p=2i+1}^{2r} (m_p + 1/2)) \cdot \\ & \prod_{j=1}^s (n_j h_{1,r} - h + \sum_{p=j+1}^s n_p + h_{1,q}) x^{h-h_{1,q}-h_{1,r}-r-\sum m_i - \sum_j n_j}, \end{aligned} \quad (25)$$

for the constant  $c_1$  (see Section 6.1 and 6.2) that depends only on  $\mathcal{Y}$ . There is a similar expression for

$$\langle w'_3, \mathcal{Y}(w_1, x) G(-m_1 - 1/2) \dots G(-m_{2r+1} - 1/2) L(-n_1) \dots L(-n_s) w_2 \rangle. \quad (26)$$

If we compare (23) with (25) (and corresponding formulas for (26)) it follows that  $Q_1(h)$  is, up to a non-zero multiplicative constant, equal to  $K_2(h_{1,r}, h_{1,q}, h)$  (singular vector is odd!) and  $Q_2(h)$  is, up to a multiplicative constant, equal to  $K_1(h_{1,r}, h_{1,q}, h)$ . ■

Thus, Proposition 6.1 and Theorem 7.1 gives us

**Theorem 7.1** (a) *As a  $A(L(3/2, 0))$ -module*

$$\begin{aligned} & A(L(3/2, h_{1,q})) \otimes_{A(L(3/2, 0))} L(3/2, h_{1,r})(0) \cong \\ & \frac{\mathbf{C}[x]}{\langle \prod_{-j \leq k \leq j} (x - h_{1,q+4k}) \rangle} \oplus \frac{\mathbf{C}[x]}{\langle \prod_{-j+1/2 \leq k \leq j+1/2} (h + 1/2 - h_{1,q+4k}) \rangle} \end{aligned} \quad (27)$$

(b) *The space*

$$I \left( \begin{array}{c} L(3/2, h) \\ L(3/2, h_{1,q}) \ M(3/2, h_{1,r}) \end{array} \right),$$

*is non-trivial if and only if  $h = h_{1,s}$  for some  $s \in \{q+r-1, q+r-3, \dots, q-r+1\}$ .*

(c) *The space*

$$I \left( \begin{array}{c} L(3/2, h) \\ L(3/2, h_{1,q}) \ L(3/2, h_{1,r}) \end{array} \right),$$

*is one-dimensional if and only if  $h = h_{1,s}$ ,  $s \in \{q+r-1, q+r-3, \dots, |q-r|+1\}$ .*

*Proof (a):* From Lemma 7.1 it follows that

$$A(L(3/2, h_{1,r})) \otimes_{A(L(3/2,0))} L(3/2, h_{1,q}) \cong \frac{\mathbb{C}[x]}{\langle Q_1(x) \rangle} \oplus \frac{\mathbb{C}[x]}{\langle Q_2(x) \rangle}. \quad (28)$$

Now we apply (18) and Proposition 6.1.

*Proof (b):* As in [M1], by examining carefully the main construction of intertwining operators in [L1] with a minor super-modifications, for every  $A(L(3/2, 0))$ -morphism from  $A(L(3/2, h_{1,q})) \otimes_{A(L(3/2,0))} L(3/2, h_{1,r})$  to  $L(3/2, h)(0)$  we can construct a non-trivial intertwining operator of the form  $I \left( \begin{smallmatrix} L(3/2, h) \\ L(3/2, h_{1,q}) \quad M(3/2, h_{1,r}) \end{smallmatrix} \right)$ .

*Proof (c):* In order to project  $\mathcal{Y} \in I \left( \begin{smallmatrix} L(3/2, h) \\ L(3/2, h_{1,q}) \quad M(3/2, h_{1,r}) \end{smallmatrix} \right)$  to a non-trivial intertwining operator of the type  $\left( \begin{smallmatrix} L(3/2, h) \\ L(3/2, h_{1,q}) \quad L(3/2, h_{1,r}) \end{smallmatrix} \right)$  (as in [M1])  $h = h_{1,s}$  for  $s \in \{q+r-1, q+r-3, \dots, q-r+1\} \cap \{r+q-1, r+q-3, \dots, r-q+1\}$ , i.e.,  $s \in \{q+r-1, q+r-3, \dots, |q-r|+1\}$ . ■

**Theorem 7.2** Suppose that  $q \geq r$  <sup>3</sup>

$$\dim I \left( \begin{smallmatrix} L(3/2, h_{1,s}) \\ L(3/2, h_{1,q}) \quad L(3/2, h_{1,r}) \end{smallmatrix} \right)_{\text{even}} = 1, \quad (29)$$

if and only if

$$s \in \{q+r-1, q+r-5, \dots, q-r+1\}$$

$$\dim I \left( \begin{smallmatrix} L(3/2, h_{1,s}) \\ L(3/2, h_{1,q}) \quad L(3/2, h_{1,r}) \end{smallmatrix} \right)_{\text{odd}} = 1 \quad (30)$$

if and only if

$$s \in \{q+r-3, q+r-7, \dots, q-r+3\}.$$

*Proof:* By using (27) we obtain the following decomposition:

$$\begin{aligned} & A^0(L(\frac{3}{2}, h_{1,q})) \otimes_{A(L(3/2,0))} L(\frac{3}{2}, h_{1,r})(0) \cong \\ & \mathbb{C}v_{q+r-1} \oplus \mathbb{C}v_{q+r-5} \dots \oplus \mathbb{C}v_{q-r+1} \\ & A^1(L(\frac{3}{2}, h_{1,q})) \otimes_{A(L(3/2,0))} L(\frac{3}{2}, h_{1,r})(0) \cong \\ & \mathbb{C}v_{q+r-3} \oplus \mathbb{C}v_{q+r-7} \dots \oplus \mathbb{C}v_{q-r+3}, \end{aligned} \quad (31)$$

where  $\mathbb{C}v_i$  is a  $\mathbb{C}[y]$ -module such that  $y.v_i = \frac{(i-1)^2}{8}v_i$ .

*Claim:* Let

$$\psi \in \text{Hom}_{A(L(c,0))}(A^0(L(\frac{3}{2}, h_{1,q})) \otimes_{A(L(3/2,0))} L(\frac{3}{2}, h_{1,r})(0), L(\frac{3}{2}, h_{1,s})(0)),$$

then the corresponding intertwining operator is even. Similarly if we start from

$$\frac{\psi \in \text{Hom}_{A(L(c,0))}(A^1(L(\frac{3}{2}, h_{1,q})) \otimes_{A(L(3/2,0))} L(\frac{3}{2}, h_{1,r})(0), L(\frac{3}{2}, h_{1,s})(0))}{^3(w_1 \quad w_2) \cong (w_2 \quad w_1)}.$$

the corresponding intertwining operator is odd.

*Proof (Claim):* Let us elaborate the proof when  $\psi$  is “even”. From the construction in [FZ] and [L2]  $\mathcal{Y}$  is obtained by lifting  $\psi$  to a mapping from  $L(3/2, h_{1,q}) \otimes L(3/2, h_{1,r})(0)$  to  $L(3/2, h_{1,s})(0)$ , such that

$$L(3/2, h_{1,q})_{\text{odd}} \otimes L(3/2, h_{1,r})(0) \mapsto 0.$$

To extend this map to a mapping from  $L(3/2, h_{1,q}) \otimes M(3/2, h_{1,r})$  to  $L(3/2, h_{1,s})$  one uses generators and PBW so the sign is preserved. The last step (projection to  $L(3/2, h_{1,q}) \otimes L(3/2, h_{1,r})$ ) is possible (because of the condition  $q \geq r$ ) so the proof follows (when  $\psi$  is odd a similar argument works). ■

Let us summarize everything.

**Corollary 7.1** *Let  $\mathcal{A}_s$  be a free abelian group with generators  $b(m)$ ,  $m \in 2\mathbb{N}+1$ . Define a binary operation  $\times : \mathcal{A}_s \times \mathcal{A}_s \rightarrow \mathcal{A}_s$ ,*

$$b(q) \times b(r) = \sum_{j \in \mathbb{N}} \dim I \left( \begin{matrix} L(3/2, h_{1,j}) \\ L(3/2, h_{1,q}) \ L(3/2, h_{1,r}) \end{matrix} \right) b(j).$$

*Then  $\mathcal{A}_s$  is a commutative associative ring, and the mapping  $b(m) \mapsto V(\frac{m-1}{4})$  gives an isomorphism to the Grothendieck ring  $\mathcal{R}ep(\mathfrak{osp}(1|2))$ .*

*Proof:* The proof follows from Theorem 7.1(c) and (4). ■

## 8 Multiplicity–2 fusion rules and super logarithmic intertwiners

### 8.1 Multiplicity–2

We have seen that in the  $c = \frac{3}{2}$  case all fusion coefficients are 0 or 1. Still, we expect (according to [HM]) that for some vertex operator superalgebras  $L(c, 0)$ , fusion coefficients are 2.

Here is one example. If  $c = 0$ , as in the case of the Virasoro algebra, the super vertex operator algebra  $L(0, 0) = \frac{M(0,0)}{\langle G(-1/2)v_0, G(-3/2)v_0 \rangle}$  is trivial. Still we can consider a vertex operator superalgebra  $V(0, 0) := \frac{M(0,0)}{\langle G(-1/2)v \rangle}$ . Clearly, for every  $h \in \mathbb{C}$ , we have (all modules are considered to be  $V(0, 0)$ –modules):

$$\dim I \left( \begin{matrix} L(0, 0) \\ L(0, h) \ L(0, h) \end{matrix} \right) = 2. \quad (32)$$

The previous example is little bit awkward. Here is a nice example with “irrational” central charge:

**Proposition 8.1**

$$\dim I \left( \begin{matrix} L(\frac{15}{2} - 3\sqrt{5}, \frac{\sqrt{5}}{2} - 1) \\ L(\frac{15}{2} - 3\sqrt{5}, \frac{3}{4}(\frac{\sqrt{5}}{2} - 1)) \ L(\frac{15}{2} - 3\sqrt{5}, \frac{3}{4}(\frac{\sqrt{5}}{2} - 1)) \end{matrix} \right) = 2. \quad (33)$$

*Proof:* It is not hard to see (by using a result from [AA] or [D]) that  $M(\frac{15}{2} - 3\sqrt{5}, \frac{3}{4}(\frac{\sqrt{5}}{2} - 1))$  has the unique submodule that is irreducible (the case  $II_+$  in [AA]). If we analyze the determinant formula [KWa], singular vectors, and then use Theorem 6.1, after some calculation we obtain (33). ■

## 8.2 Logarithmic intertwiners

In [M2] we introduced and constructed several examples of logarithmic intertwining operators. Roughly, logarithmic intertwiners exist if matrix coefficients yield some logarithmic solutions.

By straightforward super-extension we obtain the following result:

**Proposition 8.2**

$$\dim I \left( \begin{matrix} W_2(\frac{27}{2}, \frac{-3}{2}) \\ L(\frac{27}{2}, \frac{-3}{2}) \quad L(\frac{27}{2}, \frac{-3}{2}) \end{matrix} \right) = 2 \quad (34)$$

*Proof:* Again, the result follows by combining techniques from this paper and [M2] ■

## 9 Future work and open problems

- For which triples  $L(c, h_1)$ ,  $L(c, h_2)$  and  $L(c, h_3)$  do we have

$$\dim I \left( \begin{matrix} L(c, h_3) \\ L(c, h_1) \quad L(c, h_2) \end{matrix} \right) = 2?$$

- Determine the fusion ring for degenerate minimal models for  $N = 2$  superconformal algebra (cf. [M3]).
- Construct an analogue of the vertex tensor categories constructed in [HM] (by using the main result in [A]), for the models studied in this paper.

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